String spectral sequences

Jean-François LE BORGNE

February 1, 2008

Abstract

The loop homology of a closed orientable manifold M of dimension m is the commutative algebra $\mathbb{H}_*(LM) = H_{*+m}(LM)$ equipped with the Chas-Sullivan loop product. Here LM denotes the free loop on M. There is also a homomorphism of graded algebras I from $\mathbb{H}_*(LM)$ to the Pontryagin algebra $H_*(\Omega M)$. The primary purpose of this paper is to study the loop homology and the homomorphism I, when M is a fibered manifold, by mean of the associated Serre spectral sequence. Some related results are discussed, in particular the link between I and the Cohen-Jones-Yan spectral sequence.

AMS Classification : 55P35, 54N45,55N33, 17A65, 81T30, 17B55

Key words : free loop space, loop homology, Serre spectral sequence.

Introduction.

In this paper $\boldsymbol{k}$ is fixed commutative ring, (co)chain complexes, (co)homology are with coefficients in $\boldsymbol{k}$.

Let M be a 1-connected closed oriented m-manifold and let LM be its loop space. Chas and Sullivan [3] have constructed a natural product on the desuspension of the homology of the free loop space $\mathbb{H}_*(LM) := H_{*+m}(LM)$ so that $\mathbb{H}_*(LM)$ is a commutative graded algebra. This product is called the *loop product*. The purpose of this paper is to compute this algebra for m-manifolds which appear as the total space of a fiber bundle, (for example Stiefel manifolds). To be more precise let us recall that if X and Y are two Hilbert connected smooth oriented manifolds without boundary and if $f : X \to Y$ is a smooth orientation preserving embedding of codimension $k < \infty$ there is a well defined homomorphism of $H_*(Y)$-comodules

$$f_! : H_*(Y) \to H_{*-k}(X)\,,$$

called the *homology shriek map* $f_!$ (see section 2 for details).

Our main result, consists to show that if $f : X \to Y$ is a *fiber embedding* (definition below) then $f_!$ behave really nicely with associated Serre spectral sequences. The remaining results of the paper are consequences of our main result.

A fiber embedding (f, f^B) is a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & X' \\ p\downarrow & & \downarrow p' \\ B & \xrightarrow{f^B} & B' \end{array}$$

where

$$(*) \quad \begin{cases} a) & X, X', B \text{ and } B' \text{ are connected Hilbert manifolds without boundary} \\ b) & f \text{ (resp. } f^B) \text{ is a smooth embedding of finite codimension } k_X (\text{resp. } k_B) \\ c) & p \text{ and } p' \text{ are Serre fibrations} \\ d) & \text{for some } b \in B \text{ the induced map} \\ & \qquad f^F : F := p^{-1}(b) \to {p'}^{-1}(f(b)) := F' \\ & \text{is an embedding of finite codimension } k_F \\ e) & \text{embeddings } f, f^B \text{ and } f^F \text{admits Thom classes}. \end{cases}$$

First part of the main result. *Let $f : X \to X'$ be a fiber embedding as above. For each $n \geq 0$ there exist filtrations*

$$\{0\} \subset F_0 C_n(X) \subset F_1 C_n(X) \subset ... \subset F_n C_n(X) = C_n(X)$$
$$\{0\} \subset F_0 C_n(X') \subset F_1 C_n(X') \subset ... \subset F_n C_n(X') = C_n(X')$$

and a chain representative $f_! : C_*(X') \to C_{*-k_X}(X)$ of $f_! : H_*(X') \to H_{*-k_X}(X)$ satisfying:

$$f_! \left(F_* C_*(X') \right) \subset F_{*-k_B} C_{*-k_X}(X') \, .$$

The above filtrations induce the Serre spectral sequence of the fibration p and p', denoted $\{E^r[p]\}_{r\geq 0}$ and $\{E^r[p']\}_{r\geq 0}$. The chain map $f_!$ induces a homomorphism of bidegree $(-k_B, -k_F)$ between the associated spectral sequences

$$\{E^r(f_!)\} : \{E^r[p']\}_{r\geq 0} \to \{E^r[p]\}_{r\geq 0} \, .$$

Second part of the main result.
There exists a chain representative $f_!^B : C_*(B') \to C_{*-k_B}(B)$ (resp. $f_!^F : C_*(F') \to C_{*-k_F}(F)$) of $f_!^B : H_*(B') \to H_{*-k_B}(B)$ (resp. of $f_!^F : H_*(F') \to H_{*-k_F}(F)$) such that $\{E^2(f_!)\} = H_*(f_{B!}; \mathcal{H}_*(f_!^F)) : E^2_{s,t}[p'] = H_s(B'; \mathcal{H}_t(F')) \to E^2_{s-k_B, t-k_F}[p] = H_{s-k_B}(B; \mathcal{H}_{t-k_F} t(F))$, where $\mathcal{H}(-)$ denote the usual system of local coefficients.

Given a first quadrant homology spectral sequence $\{E_r\}_{r\geq 0}$ it is natural, in our context, to defined the (k_B, k_F)*-regraded spectral sequence* $\{\mathbb{E}^r\}_{r\geq 0}$ by $\mathbb{E}^r_{*,*} = E^r_{*-k_B, *-k_F}$.

Hereafter, we apply the main result in four different setting:

1. Classical intersection theory
2. Chas and Sullivan string theory concerning the free loop space LM.
3. The restricted Chas and Sullivan theory concerning the free loops on M based in a submanifold N of M.
4. The path space: M^I.

1. Classical intersection theory. Recall also that if M is m-dimensional oriented closed manifold then the desuspended homology of M :

$$\mathbb{H}_*(M) = H_{*+m}(M)$$

is a commutative graded algebra for the intersection product [1]. Since the intersection product $x \otimes y \mapsto x \bullet y$ is defined by the composition:

$$H_*(M) \otimes H_*(M) \stackrel{\times}{\to} H_*(M \times M) \stackrel{\Delta_!}{\to} H_{*-m}(M) \, ,$$

where $\Delta : M \to M \times M$ denotes the diagonal embedding and $\times$ the cross product. We deduce from the main result:

Proposition 1 *Let $N \to X \xrightarrow{p} M$ be a fiber bundle such that*

a) N, (resp. M) is a finite dimensional smooth closed oriented manifold of dimension n (resp. m)

b) M is a connected space and $\pi_1(M)$ acts trivially on $H_*(N)$.

The (m,n)-regraded Serre spectral sequence $\{\mathbb{E}^r[p]\}_{r\geq 0}$ is a multiplicative spectral sequence which congerges to the algebra $\mathbb{H}_*(X)$ and such that the tensor product of graded algebras $\mathbb{H}_*(M)\otimes\mathbb{H}_*(N)$ is a subalgebra of $\mathbb{E}^2[p]$. In particular if $H_*(M)$ is torsion free then

$$\mathbb{E}^2[p] = \mathbb{H}_*(M)\otimes\mathbb{H}_*(N)\,.$$

By Poincaré duality, we recover the multiplicative structure on the Serre spectral sequence in cohomology for the cup produit.

2. Chas and Sullivan string theory. Chas and Sullivan give a geometric construction of the loop product. An other description of the loop product is the following, [4]. One consider the diagram

$$\begin{tikzcd}
LM \arrow[d, "ev(0)"] & LM\times_M LM \arrow[l, "Comp"'] \arrow[r, "\tilde{\Delta}"] \arrow[d, "ev_\infty"] & LM\times LM \arrow[d, "ev(0)\times ev(0)"] \\
M & M \arrow[l, "="] \arrow[r, "\Delta"] & M\times M
\end{tikzcd}$$

where

a) $ev(0)(\gamma)=\gamma(0)\,,\gamma\in LM$,

b) the right hand square is a pullback diagram

c) $Comp$ denotes composition of free loops.

Here we assume that LM is a Hilbert manifolfd when we restrict to "smooth loops" and thus $(\tilde{\Delta},\Delta)$ is a fiber embedding of codimension m, [8].

The loop product $x\otimes y\mapsto x\circ y$ is defined by the composition

$$H_*(LM)^{\otimes 2}\xrightarrow{\times} H_*(LM^{\times 2})\xrightarrow{\tilde{\Delta}_!} H_{*-m}(LM\times_M LM)\xrightarrow{H_*(Comp)} H_{*-m}(LM)\,.$$

Here we assume that LM is a Hilbert manifolfd when we restrict to "smooth loops" and thus $\tilde{\Delta}$ is a fiber embedding of codimension m, [8].

The desuspended homology of LM, $\mathbb{H}_*(LM)=H_{*+m}(LM)$ is a commutative graded algebra [3].

Theorem A: The Cohen-Jones-Yan spectral sequence. *Let M be a smooth closed oriented m-manifold. The $(m,0)$-regraded Serre spectral sequence, $\{\mathbb{E}^r[ev(0)]\}_{r\geq 0}$, of the loop fibration,*

$$\Omega M\xrightarrow{i_0} LM\xrightarrow{ev(0)} M$$

is a multiplicative spectral sequence which congerges to the graded algebra $\mathbb{H}_*(LM) = H_{*+m}(LM)$ given at the E^2-level by $\mathbb{E}^2[ev(0)]=\mathbb{H}_*(M;\mathcal{H}_*(\Omega M))$ when $\mathcal{H}_*(\Omega M))$ denotes the usual system of local coefficients.

Furthermore, if we suppose that $\pi_1(M)$ acts trivially on ΩM then the tensor product of the graded algebra $\mathbb{H}_*(M)$ with the Pontrjagin algebra $H_*(\Omega M)$ is a subalgebra of the $\mathbb{E}^2[ev(0)]=\mathbb{H}_*(M;H_*(\Omega M))$.

Theorem B: The string Serre spectral sequence. *Under hypothesis of Proposition 1, the (m,n)-regraded Serre spectral sequence $\{\mathbb{E}^r[Lp]\}_{r\geq 0}$ of the Serre fibration*

$$LN\xrightarrow{Li} LX\xrightarrow{Lp} LM$$

is a multiplicative spectral sequence which converges to the algebra $\mathbb{H}_*(LX)$ and such that the tensor product of graded algebras $\mathbb{H}_*(LM)\otimes\mathbb{H}_*(LN)$ is a subalgebra of $\mathbb{E}^2[Lp]$. In particular if $H_*(LM)$ is torsion free then

$$\mathbb{E}^2[p]=\mathbb{H}_*(LM)\otimes\mathbb{H}_*(LN)\,.$$

We call this spectral sequence the string ((m,n)-regraded) Serre spectral sequence.

Theorem A is a generalization to the non 1-connected case of the result obtained in [4]. Spectral sequence in theorem B is naturally related to the spectral sequence considered in Proposition 1 by mean of each of the following diagrams

$$\begin{array}{ccccc} LN & \overset{Li}{\to} & LX & \overset{Lp}{\to} & LM \\ ev(0)^N\downarrow & & ev(0)^X\downarrow & & \downarrow ev(0)^M \\ N & \overset{Li}{\to} & X & \overset{Lp}{\to} & M \end{array}\quad,\quad \begin{array}{ccccc} LN & \overset{Li}{\to} & LX & \overset{Lp}{\to} & LM \\ \sigma^N\uparrow & & \sigma^X\uparrow & & \uparrow\sigma^M \\ N & \overset{Li}{\to} & X & \overset{Lp}{\to} & M \end{array}$$

where σ^X denotes the canonical section of $ev(0)^x$. Indeed both $ev(0)^X$ and σ^X induces homomorphisms of graded algebras between $\mathbb{H}_*(X)$ and $\mathbb{H}_*(LX)$.

3. Restricted Chas and Sullivan theory and intersection morphism. Let $i:N\hookrightarrow M$ be a smooth finite codimensionnal embedding and $\tilde{i}:L_NM=\{\gamma\in LM;\gamma(0)\in N\}\hookrightarrow LM$ be the natural inclusion. Under convenient hypothesis (see paragraph 4), $\mathbb{H}_*(L_NM)=H_{*+n}(L_NM)$ is a graded commutative algebra and Theorems A and B translate in Theorems A' and B' as stated and proved in paragraph 4.

The case $N=pt$ is particulary interresting since $L_{pt}M=\Omega M$ and the restricted homomorphism $\tilde{i}_!:\mathbb{H}_*(LM)\to H_*(\Omega M)$ is by definition the *intersection morphism* denoted I in [3]. We prove:

Proposition 3 *Let $a\in\mathbb{H}_{-m}(LM)$ be the homology class representing a point in $\mathbb{H}_*(LM)$ and $\mu_a:\mathbb{H}_*(LM)\to\mathbb{H}_{*-m}(LM)$ $x\mapsto a\circ x$ be multiplication by a. Then, $\tilde{i}_*\circ\tilde{i}_!=\tilde{i}_*\circ I=\mu_a$.*

Theorem C *Assume that for each $k\geq 0$, $H_k(\Omega M)$ is finitly generated. Then, the three following propositions are equivalent:*

(a) I is onto

(b) The differentials $\{d_n\}_{n\geq 0}$ of the Cohen-Jones-Yan spectral sequence vanish for $n\geq 2$ so that the spectral sequence collapses at the E^2-level

(c) $\tilde{i}_*$ is injective

Explicit computations are given in subsection 4.6.

4. The path space M^I. Consider the fibration $\Omega M\overset{i_1}{\to}M^I\overset{ev(0),ev(1)}{\to}M\times M$. First, one should observe that the construction of the loop product extends to M^I so that $\mathbb{H}_*(M^I)$ is a graded commutative algebra isomorphic to $\mathbb{H}_*(M)$. Secondly, there exists on $\mathbf{H}_*(M\times M)=\mathbb{H}_{*-m}(M\times M)$ a natural structure of associative algebra (not commutative and without unit). With these two datas, we prove Theorem A" which extends Theorem A (See the section 5 for more details).

Acknowledgement

We would like to thank David Chataur, Jean-Claude Thomas and Luc Mennichi for helpful comments that have considerably improved the paper.

The paper will be organized as follows. In Sect. 1 we will recall the definition of the shriek map associated to a smooth embedding. In Sect. 2 we will prove the main result. In Sect. 3 we will prove Proposition 1, Theorem A and B. In Sect. 4 we will state and prove Theorems A' and B' Proposition 2 and 3, prove Theorem C and give some examples. In Sect. 5 we will state and prove Theorems A" and Proposition 4 and give an example.

1. Recollection on Thom-Pontrjagin theory.

The purpose of this section is to precise the definition of the shriek map of an embedding [2].

1.1 Let us recall that the Thom class $\tau_{E,E'}$ of a disk bundle

$$(p) \qquad (D^k, S^{k-1}) \to (E, E') \xrightarrow{p} B$$

is an element $\tau_p \in H^k(E, E')$ whose restriction to each fiber (D^k, S^{k-1}) is a generator of $H^k(D^k, S^{k-1})$. Every disk bundle has a Thom class if $\mathbb{k} = \mathbb{F}_2$. Every oriented disk bundle has a Thom class with any ring of coefficients $\mathbb{k}$. If the disk bundle (p) has a Thom class τ then the composition

$$H_*(E, E') \stackrel{\tau_p \cap -}{\to} H_{*-d}(E) \stackrel{H_*(p)}{\to} H_{*-d}(B)$$

is an isomorphism of graded modules, called the *homology Thom isomorphism of the fiber bundle* (p).

1.2 Let N and M be two smooth closed oriented manifolds and $f : N \to M$ be a smooth orientation preserving embedding. For simplicity, we identify $f(N)$ with N. Assume further, that N is closed subspace of M and that M admits a partition of unity then there is a splitting

$$T(M)_{|N} = T(N) \oplus \nu_f$$

where ν_f is called the normal fiber bundle of the embedding. The isomorphism class of ν_f is well defined and depends only of the isotopy class of f.

Assume that the rank of $\nu_f = k$ and denote by

$$(D^k, S^{k-1}) \to (D^k_{p_f}, S^{k-1}_{p_f}) \xrightarrow{p_f} N$$

the associated disk bundle. The restriction of the exponential map induces an isomorphism Θ from $(D^k_{p_f}, S^{k-1}_{p_f})$ onto the tubular neighborhood $(\mathrm{Tube}\, f, \partial \mathrm{Tube}\, f)$.

The Thom class of the embedding f is the Thom class, whenether it exists, of the disk bundle p_f.

The definition of $f_! : H_*(M) \longrightarrow H_{*-k}(N)$ is given by the following composition:

$$H_*(M) \xrightarrow{j^M} H_*(M, M-N) \xrightarrow{Exc} H_*(\mathrm{Tube}\, f, \partial \mathrm{Tube}\, f) \xrightarrow{\Theta_*} H_*((D^k_{p_f}, S^{k-1}_{p_f}))^{p_*} \longrightarrow H_*(N)$$

where Exc denotes the excision isomorphism, $j^B : H_*(B) \to H_*(B, A)$ the canonical homorphism induced by the inclusion $A \subset B$.

2. Proof of the main result.

The aim of this section is to prove our main result. For this purpose following [4] we describe the homology shriek map at the chain level and the Serre filtration.

2.1 Shriek map of an embedding at the chain level. We use the same notations as in section 1 and we define $f_! : C_*(N) \to C_{*-k}(M)$. Let $t : M \to Tubef/\partial Tubef$ the Thom collapse map wich is identity on the interior of the tubular neighbourood $Tubef$ and collapses the rest on a single point. By construction, this map is continuous. We denote by $t_\sharp$ the map induced at the chain level by t. Chose $\tilde{\tau}_{p_f} \in C^k(D^k_{p_f}, S^{k-1}_{p_f})$ representing τ_{p_f} and let $f_!$ at the chain level be the composition:

$$C_*(M) \xrightarrow{t_\sharp} C_*(Tubef/\partial Tubef) \xrightarrow{can} C_*(Tubef, \partial Tubef) \xrightarrow{\Theta_\sharp} C_*(D^k_{p_f}, S^{k-1}_{p_f}) \xrightarrow{\psi} C_{*-k}(N)$$

where *can* is the algebraic natural application $can : C_*(A/B) \to C_*(A,B)$, $\Theta_\sharp$ the chain map induced by the exponential map and ψ defined as follow: since the cap product is well defined at the level of the cochains, we define:

$$\psi : C_*(D^k_{p_f}, S^{k-1}_{p_f}) \longrightarrow C_{*-k}(N) \qquad \alpha \mapsto (p_f)_\sharp(\tilde{\tau}_{p_f} \cap \alpha).$$

The definition of ψ depends of the choice of $\tilde{\tau}_{p_f}$ but in homology, each choice induces the same morphism: the Thom isomorphism.

2.2 Serre filtration. Following [7] there is a filtration of $C_*(X)$. $F_{-1} = 0 \subset F_0 \subset F_1 \subset ... \subset F_n \subset ...$ defined as follows: for $p \geq q$, define $(i_0, i_1, ..., i_p) : \Delta^q \longrightarrow \Delta^p$ the linear map wich maps the vertex v_k of the standard q-simplex Δ^q to the vertex v_{i_k} of Δ^p and put $F_pS_q(X) = \{\sigma \in S_*X/$ there exists $\Sigma \in S_*(B)$ such that $p \circ \sigma = \Sigma \circ (i_0, i_1, ..., i_q)\}$. The linear extension of this filtration provides a filtration of $C_*(X)$. This filtration leads to the construction of the Serre spectral sequence. To continue the proof of the main result, we need two lemmas.

2.3 Lemma 1 *A morphism of fibrations*

$$\begin{tikzcd}
F \arrow[r, "i|_F"] \arrow[d] & F' \arrow[d] \\
E \arrow[r, "i"] \arrow[d, "p"] & E' \arrow[d, "p'"] \\
B \arrow[r, "\tilde{i}"] & B'
\end{tikzcd}$$

such that $\tilde{i}(B)$ and $p'(E')$ intersect transversally in B' factorizes as

$$\begin{tikzcd}
F \arrow[r, "i|_F"] \arrow[d] & F' \arrow[r, "="] \arrow[d] & F' \arrow[d] \\
E \arrow[r, "i_{2E}"] \arrow[d, "p"] & E \times_{B'} B \arrow[r, "i_{1E}"] \arrow[d, "p''"] & E' \arrow[d, "p'"] \\
B \arrow[r, "="] & B \arrow[r, "\tilde{i}"] & B'
\end{tikzcd}$$

where p'' is the pull-back fibration of p' along i. Moreover, the upper left square is a pull-back diagram.

Proof: Consider the commutative diagram:

$$\begin{tikzcd}
F \arrow[r, dashed, "i|_F"] \arrow[d, "j_1"] & F' \arrow[r, "="] \arrow[d, "j_2"] & F' \arrow[d] \\
E \arrow[r, dotted] \arrow[dr, "p"] & E \times_{B'} B \arrow[r] \arrow[d] & E' \arrow[d, "p'"] \\
 & B \arrow[r, "\tilde{i}"] & B'
\end{tikzcd}$$

where the right hand part is a pull-back diagram and the dotted arrow is obtained by universal property. We want to show that the left hand square is a pullback diagram where j_1 and j_2 are inclusions of fibers in the total space. For this purpose we consider

the following diagram

$$\begin{tikzcd} F \arrow[r, "i|_F"] \arrow[d, "j_1"] & F' \arrow[r] \arrow[d, "j_2"] & pt \arrow[d] \\ E \arrow[r] & E' \times_{B'} B \arrow[r, "p''"] & B \end{tikzcd}$$

where pt is a point of B. The big square is a pull-back diagram. Therefore $F_1 \cong E \times_B pt \cong E \times_{(E' \times_{B'} B)} ((E' \times_{B'} B) \times_B pt)$ ($\cong$ means homeomorphic). The right square is a pull-back thus $F' \cong (E' \times_{B'} B) \times_B pt$ and $F \cong E \times_{E' \times_{B'} B} F'$.

□

2.4 Lemma 2 *Denote by ν_f the normal bundle of the embedding $f : X \hookrightarrow X'$ of the fiber embedding. If we chose $\tilde{\tau} \in C^*(D^{k_X}(\nu_f), S^{k_X-1}(\nu_f))$ such that $\tilde{\tau}$ vanishes on degenerate simplices, then*

$$F_p C_{p+q}(D^{k_X}(\nu_f), S^{k_X-1}(\nu_f)) \xrightarrow{\tilde{\tau}\cap -} F_{p-k_B} C_{p+q-k_X}(D^{k_X}(\nu_f), S^{k_X-1}(\nu_f))$$

for p,q, integers.

Proof: Applying Lemma 1 to our fiber embedding (f, f_B), we obtain the next diagram:

$$\begin{tikzcd} F \arrow[r, "f^F"] \arrow[d] & F' \arrow[r, "="] \arrow[d] & F' \arrow[d] \\ X \arrow[r, "f_2"] \arrow[d, "p"] & X \times_{B'} B \arrow[r, "f_1"] \arrow[d, "p''"] & X' \arrow[d, "p'"] \\ B \arrow[r, "="] & B \arrow[r, "f^B"] & B' \end{tikzcd} .$$

We begin by proving that right and left hand part of this diagram are fibers embedding. Since $f^B(B)$ and $p'(X')$ intersect transversally in B' [6], $X \times_{B'} B$ is a manifold so that the pull-back diagram of the right is a fiber embedding. The same holds for the left part of the diagram.

Now, we will prove lemma 2 for each fiber embedding (f_1, f^B) and (f_2, id_B).

1) Case (f_1, f^B). Let τ_{f^B} and τ_{f_1} be respectively the Thom class of f^B and f_1. By naturality, we have $\tau_{f_1} = p'^*(\tau_{f^B})$. At the chain level we can choose a cocycle representing the Thom classes, also called τ_{f^B} and τ_{f_1}, and we have $\tau_{f_1} = p'^\sharp(\tau_{f^B})$.

For some $\sigma \in F_p C_{p+q}(X')$, by definition of the Serre filtration, there exists $\Sigma \in C_p(B')$ and $(i_0, ..., i_{p+q})$ such that $p'(\sigma) = \Sigma(i_0, ..., i_{p+q})$. Thus $p'_\sharp(\tau_{f_1} \cap \sigma) = p'_\sharp(p'^\sharp(\tau_{f_B}) \cap \sigma) = \tau_{f_B} \cap \Sigma(i_0, ..., i_{p+q}) = \tau_{f_B}(i_{p+q-k_B}, ..., i_{p+q})\Sigma(i_0, ..., i_{p+q-k_B})$.

Since $\tau_{f_B}(i_{p+q-k_B}, ..., i_{p+q}) \neq 0$, $i_{p+q-k_B}, ..., i_{p+q}$ are pairwise distinct then we can write $\Sigma(i_0, ..., i_{p+q-b})$ as an element of $C_{p-k_B}(B')$ so that $f_{1!}(\sigma) \in F_{p-k_B} C_{p+q-k_B}(X' \times_{B'} B)$.

It remains to show that $f_{1!}$ preserves the differentials (the context being sufficiently clear to know about wich differential we refer, all differentials are denoted by d). Let $c \in C_*(X')$, since $d(\tau_{f_1}) = 0$ we have: $d(\tau_{f_1} \cap c) = d(\tau_{f_1}) \cap c + (-1)^{k_B} \tau_{f_1} \cap dc = (-1)^{k_B} \tau_{f_1} \cap dc$ This implies that $d(f_{1!}(c)) = (-1)^{k_B} i_!(dc)$.

2) Case (f_2, id_B). Let $\omega \in F_{p-k_B}(C_{p+q-k_B}(X' \times_{B'} B))$. Then there exists $\Omega \in C_{p-k_B}(B)$ such that $p''_\sharp(\omega) = \Omega(i_0, ..., i_{p+q-k_B})$. Let $\tau_{f_2} \in C^{k_F}(X \times'_B B)$ be a cochain representing the Thom class of the embedding f_2. Thus $\tau_{f_2} \cap \omega$ is a subchain of degree $p+q-k_X$ thus $\tau_{f_2} \cap \omega$ factorizes by Ω. Then $f_{2!}(\omega)$ lies in $F_{p-k_B} C_{p+q-k_X}(X)$. It is now easy to complete the proof. □

2.5 End of the proof of the main result. Since the Serre filtration of $C_*(X)$ is natu-

ral with respect to fiberwise maps the three first applications defining $f_!$ as in **2.1** induce morphisms of differential graduded filtred module (dgfm for short). Lemma 2 proves that the last application is a morphism of dgfm then $f_!$ induces a morphism of dgfm of bidegree (k_B, k_X).

The second part of the main result follows by classical theory of spectral sequences [7].

□

3. Proof of Proposition 1, Theorems A,B.

3.1 Proof of Proposition 1 For a topological space Y, denote by Δ_Y the diagonal embedding of Y in $Y \times Y$. The diagonal map $\Delta_X : X \longrightarrow X \times X$ factorizes so that we obtain the following commutative diagram :

$$\begin{tikzcd}
N \times N \arrow[r] & X \times X \arrow[r] & M \times M \\
N \times N \arrow[r] \arrow[u, "id"] & X \times_M X \arrow[r] \arrow[u, "i"] & M \arrow[u, "\Delta_M"'] \\
N \arrow[r] \arrow[u, "\Delta_N"] & X \arrow[r] \arrow[u, "\Delta'_X"] & M \arrow[u, "id"']
\end{tikzcd}$$

with $i \circ \Delta'_X = \Delta_X$. In particular (i, Δ_M) and (Δ'_X, id) are fiber embeddings. We apply the main result to the shriek map $\Delta_{X!}$. Since the composition

$$H_*(X) \otimes H_*(X) \xrightarrow{\times} H_*(X \times X) \xrightarrow{\Delta_{X!}} H_{*-(m+n)}(X)$$

is the intersection product one deduces that the regraded spectral sequence is a multiplicative spectral sequence with respect to the intersection product. The E^2-term is given by $\mathbb{E}^2_{p,q} = \mathbb{H}_p(M; \mathbb{H}_q(N))$ (since $\pi_1(M)$ is assumed to act trivially on $H_*(N)$ the coefficients are constant). The naturality of the cross product provides a morphism of spectral sequence given at the E^2-term by

$$\mathbb{H}_p(M; \mathbb{H}_q(N)) \otimes \mathbb{H}_{p'}(M; \mathbb{H}_{q'}(N)) \xrightarrow{\times} \mathbb{H}_{p+p'+m}(M \times M; \mathbb{H}_{q+q'+n}(N \times N))$$

Then, $\Delta_{X!}$ induces a morphism of spectral sequence given at the E^2-term by

$$\mathbb{H}_{p+p'+m}(M \times M; \mathbb{H}_{q+q'+n}(N \times N)) \xrightarrow{E^2(\Delta_{X!})} \mathbb{H}_{p+p'}(M; \mathbb{H}_{q+q'}(N))$$

such that $E^2(\Delta_{X!}) = H(\Delta_{M!}; \Delta_{N!})$. As a consequence, we find that $\mathbb{H}_*(M) \otimes \mathbb{H}_*(N) \hookrightarrow \mathbb{E}^2_{*,*}$ as subalgebra where $\mathbb{H}_*(M) \otimes \mathbb{H}_*(N)$ is the tensor product of algebra for the intersection product.

□

3.2 Proof of theorem A: the Cohen-Jones-Yan spectral sequence. In this section the results of [4] are revisited and sligtly extended. We construct the following commutative diagram:

$$\begin{tikzcd}
\Omega M \times \Omega M \arrow[r, "i \times i"] & LM \times LM \arrow[r, "ev(0) \times ev(0)"] & M \times M \\
\Omega M \times \Omega M \arrow[r, "i \times i"] \arrow[u, "id"] \arrow[d, "Comp"] & LM \times_M LM \arrow[r, "ev_\infty"] \arrow[u, "\tilde{\Delta}"] \arrow[d, "Comp"] & M \arrow[u, "\Delta"'] \arrow[d, "id"] \\
\Omega M \arrow[r, "i"] & LM \arrow[r, "ev(0)"] & M
\end{tikzcd}$$

where the map defined as in the introduction. Thus $(\tilde{\Delta}, \Delta)$ is a fiber embedding of codimension m. Now, composition of the maps

$$C_{*+d}(LM) \otimes C_{*+d}(LM) \xrightarrow{\times} C_{*+2d}(LM \times LM) \xrightarrow{\tilde{\Delta}_!} C_{*+d}(LM \times_M LM) \xrightarrow{Comp_\sharp} C_{*+d}(LM)$$

induces at the homology level Chas and Sullivan product denoted by μ. The Serre spectral sequence associated to the fibration

$$\Omega M \xrightarrow{i} LM \xrightarrow{ev(0)} M$$

satisfy $\mathbb{E}^2_{*,*} = \mathbb{H}_*(M; H_*(\Omega M))$ (here $\pi_1(M)$ acts trivialy on ΩM since M is arcwise connected).

By using the main result for $(\tilde{\Delta}, \Delta)$ and the naturality of the Serre spectral sequence for the Eilenberg-Zilbert morphism and for $\gamma_\sharp$, we show extending the result of [4] that there is a multiplicative structure on this Serre spectral sequence containing at the E^2-level the tensor product of $\mathbb{H}_*(M)$ with intersection product and $H_*(\Omega M)$ with Pontryagin product. This spectral sequence of algebra converges to $\mathbb{H}_*(LM)$.

3.3 Proof of theorem B: String Serre spectral sequence. For any topological space T, the map $map(T,E) \to map(T,B)$ is a fibration with fiber $map(T,F)$. Since $LT = map(S^1, T)$ and $LT \times_T LT = map(S^1 \vee S^1, T)$, we have the following fibrations: $LN \xrightarrow{Li} LX \xrightarrow{Lp} LM$ and $LN \times_N LF \xrightarrow{Li} LX \times_X LX \xrightarrow{Lp} LM \times_M LM$ Consider the pull-back diagrams

$$\begin{array}{ccc} X \times_M X & \longrightarrow & X \times X \\ \downarrow & & \downarrow{\scriptstyle p\times p} \\ M & \xrightarrow{\Delta_M} & M \times M \end{array} \quad \text{and} \quad \begin{array}{ccc} LX \times_{X\times_M X} LX & \xrightarrow{\Delta_1} & LX \times LX \\ \downarrow{\scriptstyle Lp\times_{X\times_M X} Lp} & & \downarrow{\scriptstyle Lp\times Lp} \\ LM \times_M LM & \xrightarrow{\Delta_M} & LM \times LM. \end{array}$$

with $LX \times_{X\times_M X} LX = \{(\gamma_1, \gamma_2) \in LX \times LX \quad ; \quad p(\gamma_1(0)) = p(\gamma_2(0))\}$. We deduce the following commutative diagram:

$$\begin{array}{ccccc} LN \times LN & \longrightarrow & LX \times LX & \longrightarrow & LM \times LM \\ \uparrow{\scriptstyle id} & & \uparrow{\scriptstyle i} & & \uparrow{\scriptstyle L\Delta_M} \\ LN \times LN & \longrightarrow & LX \times_{X\times_M X} LX & \longrightarrow & LM \times_M LM \\ \uparrow{\scriptstyle L\Delta_N} & & \uparrow{\scriptstyle L\Delta'_X} & & \uparrow{\scriptstyle id} \\ LN \times_N LN & \longrightarrow & LX \times_X LX & \longrightarrow & LM \times_M LM \\ \downarrow{\scriptstyle Comp_N} & & \downarrow{\scriptstyle Comp_X} & & \downarrow{\scriptstyle Comp_M} \\ LN & \longrightarrow & LX & \longrightarrow & LM \end{array}$$

with obviously defined map. From the definition of μ_X, naturality and the main result, we construct a product on the Serre spectral sequence associated to the fibration $LN \xrightarrow{Lj} LX \xrightarrow{Lp} LM$ To achieve the proof, we need:

Lemma *Let* $F \xrightarrow{j} E \xrightarrow{p} B$ *be a fibration such that* $\pi_1(B)$ *acts trivially on* $H_*(F)$, *then* $\pi_1(LB)$ *acts trivially on* $H_*(LF)$.

Proof of the lemma: For a fixed loop $\gamma \in \Omega B$, the holonomy operation defines maps

$$\Psi_\gamma : F \longrightarrow F \qquad x \mapsto \gamma.x.$$

Since $\pi_1(B)$ acts trivially on $H_*(F)$, there exists a homotopy $H_\gamma : F\times I \longrightarrow F \qquad (x,t) \mapsto H_\gamma(x,t)$ such that $H_\gamma(x,0) = x$, and $H_\gamma(x,1) = \gamma.x$. For a fixed $\Gamma \in \Omega LB$, the holonomy action of ΩLB on LF (associated to the fibration $LF \xrightarrow{Lj} LE \xrightarrow{Lp} LB$) yields

$$\Phi_\gamma : LF \longrightarrow LF \qquad f \mapsto (s \mapsto \Psi_{\Gamma(-,s)}(f(s)))$$

Now the homotopy

$$\mathcal{H}_\Gamma : LF \times I \longrightarrow LF \qquad f(-), t \mapsto \mathcal{H}_\Gamma(f(-),t) = (s \mapsto \mathcal{H}_\Gamma(-,s)(f(s),t))$$

satisfies

$$\mathcal{H}_\Gamma(f,0) = (s \mapsto H_{\Gamma(-,s)}(f(s),0)) = f(-))$$
$$\mathcal{H}_\Gamma(f,1) = (s \mapsto H_{\Gamma(-,s)}(f(s),1)) = \Gamma(-,-).f(-) = \Phi_{\Gamma.f}$$

$s \in S^1$.

□

The above lemma proves that the local coefficients in the spectral sequence are constant. Furthermore, $\mathbb{E}^2_{*,*} = H_{*+m}(LM; H_{*+n}(LN))$ contains $\mathbb{H}_*(LM) \otimes \mathbb{H}(LN)$ as subalgebra.

□

4. Restricted Chas and Sullivan algebra and intersection morphism. Proof of theorem A',B',C and proposition 2, 3.

4.1 The restricted Chas and Sullivan loop product. Let $i : N \hookrightarrow M$ be a smooth finite codimensionnal embedding of Hilbert closed manifold such that:

*) M (resp. N) is of finite dimension m (resp. n).

**) the embedding i admit a Thom class.

Define L_NM as the right corner of the pull-back diagram:

$$\begin{tikzcd} L_NM \arrow[r, "\tilde{i}"] \arrow[d, "ev(0)"] & LM \arrow[d, "ev(0)"] \\ N \arrow[r, "i"] & M \end{tikzcd}$$

i.e L_NM is the space of free loop spaces of LM based on N. Denote $\mathbb{H}_*(L_NM) := H_{*+n}(L_NM)$. We define the restricted loop product

$$\mu_N : \mathbb{H}_*(L_NM) \otimes \mathbb{H}_*(L_NM) \rightarrow \mathbb{H}_*(L_NM)$$

$$\alpha \otimes \beta \mapsto \mu_N(\alpha \otimes \beta) := \alpha \circ_N \beta$$

by putting $\mu_N = Comp_* \circ \Delta_{N!} \circ \times$ where we consider the commutative diagram:

$$\begin{tikzcd} L_NM \arrow[d, "ev(0)"] & L_NM \times_N L_NM \arrow[l, "Comp"'] \arrow[r, "\tilde{\Delta_N}"] \arrow[d, "ev_\infty"] & L_NM \times_N L_NM \arrow[d, "ev(0)\times ev(0)"] \\ N & N \arrow[l, "="] \arrow[r, "\Delta_N"] & N \times N \end{tikzcd}$$

Theorem A' *The (n,0)-regraded spectral sequence associated to the fibration $\Omega M \longrightarrow L_NM \longrightarrow N$ is multiplicative. Moreover, if $\pi_1(N)$ acts trivially on ΩM, the E^2-term of the spectral sequence contains $i_!(\mathbb{H}_*(M)) \otimes H_*(\Omega M)$ as subalgebra.*

Proof: Starting with the following commutative diagram:

$$\begin{tikzcd}
\Omega M \times \Omega M \arrow[r] & L_N M \times L_N M \arrow[r, "ev(0)\times ev(0)"] & N \times N \\
\Omega M \times \Omega M \arrow[u, "id"] \arrow[r] \arrow[d, "Comp"] & L_N M \times_N L_N M \arrow[u, "\tilde{\Delta_N}"] \arrow[r, "ev_\infty"] \arrow[d, "Comp"] & N \arrow[u, "\Delta_N"'] \arrow[d, "id"] \\
\Omega M \arrow[r] & L_N M \arrow[r, "ev(0)"] & N
\end{tikzcd}$$

the proof of Theorem A works as well to prove Theorem A'.

□

Proposition 2 *$\tilde{i}_!$ is a morphism of algebra and induces a morphism of multiplicative spectral sequence: $E^*(\tilde{i}_!) : \mathbb{E}^*(LM) \longrightarrow \mathbb{E}^*(L_N M)$.*

Proof: Observe first that $(\tilde{i}, i)$ is a fiber embedding so that the result comes immediately from the main result and from the following commutative diagram:

$$\begin{tikzcd}
L_N M \times L_N M \arrow[r, "\tilde{i}\times\tilde{i}"] & LM \times LM \ . \\
L_N M \times_N L_N M \arrow[u] \arrow[r, "\tilde{i}\times_N\tilde{i}"] \arrow[d, "Comp"] & LM \times_M LM \arrow[u] \arrow[d, "Comp"] \\
L_N M \arrow[r, "\tilde{i}"] & LM
\end{tikzcd}$$

Indeed $\tilde{i}_! \circ \mu = \mu_N \circ \tilde{i}_! \otimes \tilde{i}_!$.

□

Now, let state theorem B'. Let $N \xrightarrow{i} X \xrightarrow{p} M$ be the fibration of theorem B and $j_V : V \hookrightarrow M$ an embedding satisfying conditions (*) and (**). Construct the induced bundle $p_{|Y}$ from the pull-back diagram:

$$\begin{tikzcd}
U \arrow[r, "j_{|U}"] \arrow[d, "i_{|U}"'] & N \arrow[d, "i"] \\
Y \arrow[r, "j"] \arrow[d, "p_{|Y}"] & X \arrow[d, "p"] \\
V \arrow[r, "j_V"] & M
\end{tikzcd}$$

Theorem B' *In the above situation there is a morphism of multiplicative spectral sequence $\mathbb{E}^*_{*,*}(Lp) \xrightarrow{E(\tilde{j}_!)} \mathbb{E}^*_{*,*}(p_{|Y})$ given at the E^2-level by $E(\tilde{j}_!) = H_*(j_{V!}; j_{|U}!)$.*

Proof: The pull-back diagram:

$$\begin{tikzcd}
L_U N \arrow[r, "\tilde{j_{|U}}"] \arrow[d, "Li_{|U}"'] & LN \arrow[d, "Li"] \\
L_Y N \arrow[r, "\tilde{j}"] \arrow[d, "Lp_{|Y}"] & LX \arrow[d, "Lp"] \\
L_V N \arrow[r, "\tilde{j_V}"] & LM
\end{tikzcd}$$

is a fiber embedding. The theorem comes directly from the main result.

□

Remark: If $M = N \times U$, and if $p : M \longrightarrow N$ is the projection on the first factor and if $i : N \hookrightarrow M$ is a standard embedding, then $p_* \circ \tilde{i}_! : \mathbb{H}_*(LM) \longrightarrow \mathbb{H}_*(L_N M) \longrightarrow \mathbb{H}_*(LN)$ is the projection $\mathbb{H}_*(LN) \otimes \mathbb{H}_*(U) \longrightarrow \mathbb{H}_*(N)$.

4.3 Proof of proposition 3. We consider the following commutative diagram:

$$\begin{tikzcd}
pt \times LM \arrow[r, "j"] & LM \times LM \\
pt \times \Omega M \arrow[r] \arrow[u, "id\times\tilde{i}"] \arrow[d, "Comp"] & LM \times_M LM \arrow[u, "\tilde{\Delta}"'] \arrow[d, "Comp"] \\
\Omega M \arrow[r, "\tilde{i}"] & LM
\end{tikzcd}$$

We observe that $\tilde{\Delta}_{|_{pt\times\Omega M}} = id \times \tilde{i}$ and that $Comp : pt \times \Omega M \to \Omega M$ is homotopic to the identity. Denote by EZ the cross product. The map $\mathbb{H}_*(LM) \to \mathbb{H}_{*-m}(LM)$ $x \mapsto Comp_* \circ \tilde{\Delta}_! \circ EZ(pt, x)$ is in fact multiplication by a, with a the homology class of pt in $\mathbb{H}_{*-m}(LM)$. The other map $\tilde{i}_* \circ Comp_* \circ (id \times \tilde{i}_!)EZ(pt, -)$ is equal to $\tilde{i}_* \circ \tilde{i}_!$.

□

4.4 Beginning of the proof of theorem C. (1) Assume that for $n \geq 2$, all the differentials of the Cohen-Jones-Yan spectral sequence vanish, then the homomorphism $E^*_{*,*}(\tilde{i}_!)$ induced by the fiber embedding

$$\begin{tikzcd}
\Omega M \arrow[r, hook, "id"] \arrow[d, "id"] & \Omega M \arrow[d] \\
\Omega M \arrow[r, hook, "\tilde{i}"] \arrow[d, "ev(0)"] & LM \arrow[d, "ev(0)"] \\
pt \arrow[r, hook, "i"] & M
\end{tikzcd}$$

is clearly onto. Denote by $E^*_{*,*}(1)$ the spectral sequence associated to the left fibration and by $\mathbb{E}^*_{*,*}(2)$ the $(d, 0)$-regraded spectral sequence associated to the right fibration. Then, at the aboutment, $E^\infty(\tilde{i}_!)$ is onto on the graded space of $H_*(\Omega M)$ then I is onto.

(2) We begin by proving that all the differentials starting from $\mathbb{E}^*_{0,*}(2)$ vanish. We write the naturality of the Serre spectral sequence to the shriek map of the fiber embedding $(\tilde{i}, i)$ shown in the main result. We have the following commutative diagram:

$$\begin{tikzcd}
H_*(\Omega M) \simeq \mathbb{E}^2_{0,*}(2) \arrow[d, "E^2(\tilde{i}_!)=i_!\otimes id"] & \supseteq \mathbb{E}^3_{0,*}(2) & \supseteq ... & \supseteq \mathbb{E}^\infty_{0,*}(2) \arrow[d, "E^\infty(\tilde{i}_!)"] & \mathbb{H}_*(LM) \arrow[l] \arrow[dl, "\tilde{i}_!"] \\
H_*(\Omega M) \simeq E^2_{0,*}(1) & = E^3_{0,*}(1) & = ... & = E^\infty_{0,*}(1) = H_*(\Omega M) &
\end{tikzcd}$$

If $I = \tilde{i}_!$ is onto and since each $H_k(\Omega M)$ is finitely generated, we have

$$\mathbb{E}^2_{0,*}(2) = \mathbb{E}^3_{0,*}(2) = ... = \mathbb{E}^\infty_{0,*}(2).$$

Thus the differentials starting from $\mathbb{E}^*_{0,*}(2)$ vanish. We remark that in this case, the morphism at the top right of the above diagram is in fact I. The existence of the canonical section $M \to LM$ implies that the differentials starting from $\mathbb{E}^*_{*,0}$ vanish. The multiplicative structure of $\mathbb{E}^2_{*,*}(2)$ implies that all the differentials of the Cohen-Jones-Yan spectral sequence vanish.

To end the proof of Theorem C we need:

Lemma *If one of the differential of the Cohen-Jones-Yan spectral sequence is non zero, then there exists a non-zero differential arriving on* $\mathbb{E}^*_{-d,*}(2)$.

Proof : Denote by $[M] \in \mathbb{H}_*(M)$ the fundamental class of M and by 1_ω the unit of $H_*(\Omega M)$. Assume that there exist a non-zero differential in the Cohen-Jones-Yan spectral sequence. We consider the first page of the spectral sequence where there is a non-zero differential. This page is isomorphic to $\mathbb{E}^2_{*,*}$ as an algebra. Since the Cohen-Jones-Yan spectral sequence is multiplicative, there exists a non-zero differential starting from a generator of $\mathbb{E}^2_{*,*}$ namely an element $[M] \otimes \omega$ of $\mathbb{E}^2_{0,*}$. Let $x \otimes \omega' = d([M] \otimes \omega)$ and $y \in \mathbb{H}_*(M)$ such that $x \bullet y = *$ with $* \in \mathbb{H}_{-d}(M)$ representing a fixed point. Then, $d(y \otimes \omega) = \pm d(y \otimes 1_\omega \circ [M] \otimes \omega) = \pm y \otimes 1_\omega \circ d([M] \otimes \omega) = \pm y \otimes 1_\omega \circ x \otimes \omega' = \pm x \bullet y \otimes \omega' = \pm * \otimes \omega'$. Moreover $d(y \otimes 1_\omega) = 0$ because of the existence of a section.

□

4.5 End of proof of theorem C.

Observation If $f : M \to N$ is a map between Poincaré duality manifolds, then $f_!$ is onto iff f_* injective. We prove that this result is true for the embedding $\tilde{i} : \Omega M \hookrightarrow LM$.

Proof: (1) Assume that for $n \geq 2$, all the differentials of the Cohen-Jones-Yan spectral sequence vanish, then the homomorphism $E^*_{*,*}(\tilde{i}_*)$ is clearly injective. Furthermore, we have the injective map $\mathbb{E}^\infty_{-d,*}(2) \hookrightarrow \mathbb{H}_*(LM)$. The composition of this two application is $\tilde{i}_*$.

(2) Assume $\tilde{i}_*$ is injective. From the naturality of the Serre spectral sequence for $\tilde{i}_*$, we deduce the following commutative diagram:

$$\begin{tikzcd}
H_*(\Omega M) \simeq E^2_{0,*}(1) \arrow[d, "E^2(\tilde{i}_*)"] & = E^3_{0,*}(1) & = \ldots & = E^\infty_{0,*}(1) \arrow[d, "E^\infty(\tilde{i})"] & = H_*(\Omega M)\ . \arrow[d, "\tilde{i}"] \\
H_*(\Omega M) \simeq \mathbb{E}^2_{-d,*}(2) \arrow[r, two heads] & \mathbb{E}^3_{0,*}(2) \arrow[r, two heads] & \cdots \arrow[r, two heads] & \mathbb{E}^\infty_{0,*}(2) \arrow[r, hook] & \mathbb{H}_*(LM)
\end{tikzcd}$$

Since $\tilde{i}_*$ is injective and $H_k(\Omega M)$, $H_k(LM)$ are of finite type, the surjective maps at the bottom of the diagram are in fact equalities. This proves that there is no non-zero differentials arriving on $\mathbb{E}^*_{-d,*}(2)$. We conclude with the above lemma that all the differentials of the Cohen-Jones-Yan spectral sequence are zero.

□

4.6 Spheres. *Assume* $n \geq 2$.

$I : \mathbb{H}_k(LS^{2n-1}) \to H_k(\Omega S^{2n-1})$ *is an isomorphism for* $k = 2ni$ $i \geq 0$, 0 *otherwise.*

$I : \mathbb{H}_k(LS^{2n}) \to H_k(\Omega S^{2n})$ *is an isomorphism for* $k = 2i(2n-1)$ $i \geq 0$, 0 *otherwise.*

Proof: In [4], Cohen Jones and Yan have shown that all the differentials of their spectral sequence are zero for odd spheres. Then Theorem C proves that $im(I) = H_*(\Omega S^{2n+1})$.

For the case of even dimensionnal spheres (except the 2-sphere), we need the results of [5] wich proves the following result with rational coefficients: $im(I) = H_k(\Omega S^{2n}; \mathbb{Q})$ $k = 2i(n-1)$, 0 elsewhere. This result gives the image of the torsion free part of I. For degree reasons, the image of the torsion part is zero.

□

4.7 Stiefel manifolds. Consider the fibration $S^5 \to SO(7)/SO(5) \to S^6$. Using together Theorems A and B, we prove that the differentials of the Cohen-Jones-Yan spectral sequence are zero until level 2 while the extension issues are not trivial. Applying theorem C, we obtain that $im(I) = H_*(\Omega(SO(7)/SO(5))) = \mathbb{Z}[a] \otimes \mathbb{Z}_2[b]$ with $deg(a) = 2(6-1) = 10$ and $deg(b) = 5 - 1 = 4$.

5. Application of the main result to the space of free paths.

5.1 The last application uses the composition product on the space of free paths of M, denoted by M^I. More explicitely, for two paths $\gamma_1, \gamma_2 \in M^I$ such that $\gamma_1(1) = \gamma_2(0)$,

we denote by $\gamma_1 * \gamma_2$ the composed path. On homology, we define the *path product* $\tilde{\mu} = Comp_* \circ \hat{i}_! : H_*(M^I) \otimes H_*(M^I) \longrightarrow H_{*-m}(M^I)$ from the following diagram:

$M^I \times M^I \xleftarrow{\hat{i}} M^I \times_{M^3} M^I \xrightarrow{Comp} M^I$ where $\hat{i}$ is the canonical inclusion.

5.2 Proposition 4 *The path product on $\mathbb{H}_*(M^I)$ is identified, via the isomorphism $H_*(M^I) \simeq H_*(M)$ to the intersection product on $\mathbb{H}_*(M)$.*

Proof: The homotopy $\mathcal{H} : LM \times I \longrightarrow M^I \quad \gamma, s \longmapsto (t \mapsto \gamma(st))$ proves that the inclusion $j : LM \hookrightarrow M^I$ is homotopic to the projection $ev(0) : LM \longrightarrow M$. Since composition of paths on M^I restricted to LM is the composition of loops, the following diagram is commutative:

$$\begin{tikzcd}
\mathbb{H}_*(M^I) \otimes \mathbb{H}_*(M^I) \arrow[r, leftrightarrow] \arrow[d, "\tilde{\mu}"] & \mathbb{H}_*(M) \otimes \mathbb{H}_*(M) \arrow[d, "\bullet"'] & \mathbb{H}_*(LM) \otimes \mathbb{H}_*(LM) \arrow[l, "ev(0)\otimes ev(0)"'] \arrow[d, "\circ"] \\
\mathbb{H}_*(M^I) \arrow[r, leftrightarrow] & \mathbb{H}_*(M) & \mathbb{H}_*(M^I) \arrow[l, "ev(0)"']
\end{tikzcd}$$

where $\circ$ is the Chas and Sullivan loop product, $\bullet$ is the intersection product and $\tilde{\mu}$ is the path product on $\mathbb{H}_*(M^I)$. This ends the proof of Proposition 2.

$\square$

5.3 Denote $\mathbf{H}_*(M \times M) = H_{*+d}(M \times M) = \mathbb{H}_{*-d}(M \times M)$. Define the new product

$$\diamond : \mathbf{H}_*(M \times M) \otimes \mathbf{H}_*(M \times M) \longrightarrow \mathbf{H}_*(M \times M)$$

$$(a \times b) \otimes (c \times d) \longmapsto p_{2*}(a \times (b \bullet c) \times d)$$

where $p_2 : M \times M \times M \longrightarrow M \times M$ is the projection on the first and the third factor of $M \times M \times M$. This product is associative, not commutative without unit (cf example at the end).

5.4 Theorem A" *Let M be a smooth closed m-dimensional oriented manifold. There is a multiplicative structure on the $(d, 0)$-regraded Serre spectral sequence associated to the fibration $\Omega M \longrightarrow M^I \xrightarrow{(ev_1, ev_0)} M \times M$. Furthermore, if we suppose that $\pi_1(M)$ acts trivially on ΩM then we have at the E^2-level:*

$\mathbf{E}^2_{*,*} = \mathbf{H}_*(M \times M; H_*(\Omega M))$ contains $\mathbf{H}_*(M \times M) \otimes H_*(\Omega M)$ as subalgebra. The structure of algebra on $\mathbf{H}_*(M \times M) \otimes H_*(\Omega M)$ is given by $\diamond$ on $\mathbf{H}_*(M \times M)$ and by the Pontryagin product on $H_*(\Omega M)$. Furthermore, we have $\mathbf{E}^2_{*,*} \Rightarrow \mathbb{H}_*(M)$ as algebra for the intersection product.

Proof: Consider the following commutative diagram:

$$\begin{tikzcd}
\Omega M \times \Omega M \arrow[r] & M^I \times M^I \arrow[r, "(ev(0),ev(1))\times(ev(0),ev(1))"] & M \times M \times M \times M \\
\Omega M \times \Omega M \arrow[r] \arrow[u, "id"] \arrow[d, "Comp"] & M^I \times_{M^3} M^I \arrow[r] \arrow[u, hook, "\tilde{D}"] \arrow[d, "Comp"] & M \times M \times M \arrow[u, hook, "D"] \arrow[d, "p_2"] \\
\Omega M \arrow[r] & M^I \arrow[r, "(ev(0),ev(1))"] & M \times M
\end{tikzcd}$$

where $D : M^3 \longrightarrow M^4 \quad , \quad (x, y, z) \longmapsto (x, y, y, z)$ and $\tilde{D}$ is defined from D by pull-back. Let us denote γ the composition of paths or pointed loops. The upper part of the diagram is a fiber embedding, so that, applying the main result, $\tilde{D}_!$ induces the morphism of Serre spectral sequences $E^*_{*,*}(\tilde{D}_!)$. The lower part of the diagram is a morphism of fibration. Then $E^*_{*,*}(Comp_* \circ \tilde{D}_!) : \mathbf{E}^*_{*,*}(M^I \times M^I) \longrightarrow \mathbf{E}^*_{*,*}(M^I)$ is a morphism of spectral sequence

witch provides the annonced multiplicativity of the Serre spectral sequence associated to the fibration $\Omega M \longrightarrow M^I \longrightarrow M \times M$.

$\square$

5.5 Remark In the case of a fibred space, we could state the Theorem B" analogous of Theorems B and B' , but it is not necessary since, by Proposition 4, this Theorem B" is in fact Proposition 1.

5.6 Example: the $\diamond$ product on $\mathbf{H}_*(S^3 \times S^3)$.

We apply the above result to the fibration $\Omega S^3 \longrightarrow S^{3^I} \xrightarrow{(ev(1),ev(0))} S^3 \times S^3$. Denote by $[S^3] \in H_3(S^3)$ the fundamental class and by 1 a generator of $H_0(S^3)$. Then, we obtain the following "table of multiplication" for $\diamond$:

$\diamond$	1×1	$[S^3] \times 1$	$1 \times [S^3]$	$[S^3] \times [S^3]$
1×1	0	1×1	0	$1 \times [S^3]$
$[S^3] \times 1$	0	$[S^3] \times 1$	0	$[S^3] \times [S^3]$
$1 \times [S^3]$	1×1	0	$1 \times [S^3]$	0
$[S^3] \times [S^3]$	$1 \times [S^3]$	0	$[S^3] \times [S^3]$	0

Here we have: $\mathbf{E}^2_{*,*} = \mathbf{E}^2_{-3,*} \oplus \mathbf{E}^2_{0,*} \oplus \mathbf{E}^2_{3,*}$ We denote by 1_Ω a generator of $H_0(\Omega S^3)$ and by u a generator of $H_2(\Omega S^3)$. We put $a = (1 \times [S^3] + [S^3] \times 1) \otimes 1_\Omega$, $b = (1 \times [S^3] - [S^3] \times 1) \otimes 1_\Omega$ and $c = ([S^3] \times [S^3]) \otimes 1_\Omega$. The only non zero differential is d_3 and we have $d_3(a) = 0$, $d_3(b) = (1 \times 1) \otimes 1_\Omega$ and $d_3(c) \neq 0$ lies in $\mathbf{E}^3_{0,2}$. At the aboutment, it remains only $(1 \times 1) \otimes 1_\Omega$ representing 1 in $\mathbb{H}_{-3}(S^3)$ and a representing $[S^3]$ in $\mathbb{H}_0(S^3)$. Let us denote by $\circ$ the induced product on the shifted spectral sequence. We check that $a \circ a = (1 \times [S^3] + [S^3] \times 1) \otimes 1_\Omega \quad \circ \quad (1 \times [S^3] + [S^3] \times 1) \otimes 1_\Omega = (1 \times [S^3] + [S^3] \times 1) \diamond (1 \times [S^3] + [S^3] \times 1) \otimes (1_\Omega * 1_\Omega) = (1 \times [S^3] + [S^3] \times 1) \otimes 1_\Omega = a$ wich correspond to $[S^3] \bullet [S^3] = [S^3]$ in $\mathbb{H}_*(S^3)$. In the same way, we check that $(1 \times 1) \otimes 1_\Omega \quad \circ \quad (1 \times 1) \otimes 1_\Omega = 0$ and that $(1 \times 1) \otimes 1_\Omega \quad \circ \quad (1 \times [S^3] + [S^3] \times 1) \otimes 1_\Omega = (1 \times [S^3] + [S^3] \times 1) \otimes 1_\Omega$ and $(1 \times [S^3] + [S^3] \times 1) \otimes 1_\Omega \quad \circ \quad (1 \times 1) \otimes 1_\Omega = (1 \times [S^3] + [S^3] \times 1) \otimes 1_\Omega$. Thus we recover the intersection product on $\mathbb{H}_*(S^3)$.

5.7 Remark As a final remark, let us consider the fiber embedding:

$$\begin{tikzcd}
\Omega M \arrow[r, "id"] \arrow[d] & \Omega M \arrow[d] \\
LM \arrow[r, hook, "\tilde{\Delta}"] \arrow[d, "ev(0)"] & M^I \arrow[d, "ev(0) \times ev(1)"] \\
M \arrow[r, hook, "\Delta"] & M \times M
\end{tikzcd}$$

(Δ is the diagonal embedding). Applying the main result, there is a morphism of spectral sequences: $E(\tilde{\Delta}_!) : \mathbf{E}^*_{*,*} \longrightarrow \mathbb{E}^*_{*-d,*}$ given at the E^2-level by: $E^2(\tilde{\Delta}_!) : \mathbf{E}^2_{*,*} = \mathbf{H}_*(M \times M; \mathcal{H}_*(\Omega M)) \longrightarrow \mathbb{H}_{*-d}(M; \mathcal{H}_*(\Omega M))\,, \quad (x \times y; \omega) \longmapsto (x \bullet y; \omega)$. This morphism of spectral sequences is not multiplicative.

References

[1] G. E. Bredon, *Topology and geometry*, Springer-Verlag 1993

[2] D. Chataur and J-C. Thomas *Quasi-Frobenius rationnal loop algebra*, preprint math.AT/0407014.

[3] M. Chas and D. Sullivan, *String topology*, preprint math.GT/9911159 To appear in Annals of Math.

[4] R. Cohen, J.D.S. Jones and J. Yan, *The loop homology of spheres and projective spaces.*, in Categorical Decomposition Techniques in Algebraic Topology. Progress in Mathematics **215** - Birkhauser-Verlag - 2004

[5] Y.Félix, J-C Thomas and M.Vigué-Poirrier *The Hochschild cohomology of a closed manifold* Publication mathématiques de l'IHES 99 (2004) 235-252

[6] S. Lang *Differential and Riemannian manifolds*, Springer 1995

[7] J. Mac Cleary, *Users guide to spectral sequences*, 2nd edition, Cambridge 2001.

[8] S.A. Merkulov, *De Rham model for string topology*, preprint, math.AT/0309038 (2003).